

## Weil Algebras and Supersymmetry

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We give a new interpretation for the super loop space that has been used to formulate supersymmetry. The fermionic coordinates in the super loop space are identified as the odd generators of the Weil algebra. Their bosonic superpartners are the auxiliary fields. The general  $N = 1$  supermultiplet is interpreted in terms of Weil algebras. As specific examples we consider supersymmetric quantum mechanics, Wess-Zumino model and supersymmetric Yang-Mills theory in four dimensions. Some comments on the formulation of constrained systems and integrable models and non-Abelian localization are given.

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# 1 Introduction

Equivariant cohomology (for a review see eg. [1]) arises when a group acts on a manifold. It yields a geometrical characterization of the dynamics of various systems, eg. supersymmetric and topological field theories. Mathai-Quillen [2] and Duistermaat-Heckman [3] localization schemes also utilize equivariant cohomology. In topological theories the BRST-operator has the structure of an equivariant exterior derivative and the observables are equivariant cohomology classes [4, 5]. In supersymmetric theories the supersymmetry is generated by an equivariant derivative associated to a circle action in a loop space [6]. The localization theorems, both finite dimensional and path integral generalizations, also utilize the equivariant cohomology.

In this article we shall discuss different models for equivariant cohomology and especially the the formulation of supersymmetric theories. To do this we introduce the Weil algebra related to the group acting on the manifold. With the Weil algebra we can formulate the Cartan, Weil and BRST-models [7]. The fields of the theories are divided to two classes. Half of the fields are interpreted as the generators of the exterior algebra on the loop space over function space. The other half generates the loop space over the Weil algebra associated to the group action. This distinction gives another geometric interpretation of supersymmetry different from the super loop space construction [6].

This article is organized as follows. We first represent models for equivariant cohomology. The central concept is the Weil algebra which is used to define the equivariant differentials. Then we show that by adding a contractions along the circle action generators in the relevant loop spaces we obtain supersymmetry operators. This equivariant structure is shown to exist in a general  $N = 1$  supersymmetric theory by formulating the transformation rules of the general  $N = 1$  multiplet [9] with the equivariant operators. As specific examples we discuss supersymmetric quantum mechanics and four dimensional Wess-Zumino and super Yang-Mills theories.

# 2 Equivariant Cohomology and Supersymmetry

Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$  acting on a manifold  $M$ . The action of  $G$  is generated by vector fields  $\{\chi_a, a = 1, \dots, \dim G\}$  obeying the commutation relations of  $\mathfrak{g}$

$$[\chi_a, \chi_b] = f_{ab}^c \chi_c. \quad (1)$$

Here we adopt the notation that group indices are from the beginning and the space indices from the middle of the alphabet. The corresponding Lie derivatives  $\mathcal{L}_a = [d, \iota_a]_+$  obey the same algebra. Here  $\iota_a$  denotes the contraction along the Hamiltonian vector field  $\chi_a^k = \omega^{kl} \partial_l H_a$ . If the  $G$ -action is symplectic so that it preserves the symplectic 2-form,  $\mathcal{L}_a \omega = d\iota_a \omega = 0$  and  $H^1(M) = 0$  we can introduce a momentum map  $H(x) :$

$M \rightarrow \mathfrak{g}^*$  [8] such that when one evaluates it on a given Lie algebra element  $\xi$  it yields the corresponding Hamiltonian on  $M$

$$\langle H(x), \xi \rangle = H_\xi(x) . \quad (2)$$

This provides a one-to-one correspondence between vector fields  $\chi_a$  and corresponding components  $H_a$  of the momentum map  $H = \phi^a H_a$  where  $\{\phi^a\}$  is the symmetric basis of  $\mathfrak{g}^*$ . The functions  $H_a$  satisfy the Poisson algebra

$$\{H_a, H_b\} = \partial_k H_a \omega^{kl} \partial_l H_b = f_{ab}^c H_c + \kappa_{ab} \quad (3)$$

where  $\kappa_{ab}$  is a possible Lie algebra 2-cocycle. Equivariant cohomology is roughly the de Rham cohomology of the quotient space  $M/G$ . However, when  $G$  does not act freely (it has fixed points),  $M/G$  is not a manifold. Therefore, a more precise definition is required.

The topological definition uses universal bundles. The universal bundle  $EG \rightarrow BG$  is such a bundle that every  $G$ -principal bundle over  $M$  is obtained by a pull-back:  $P(G) = f^*(EG)$  for some  $f : M \rightarrow BG$ . The universal bundle  $EG$  is a contractible manifold with a free  $G$ -action and thus we can form the associated bundle  $EG \times_G M$ . The topological equivariant cohomology is the de Rham cohomology of  $EG \times_G M/G$ .

In this article we are more interested in the algebraic equivariant cohomology since we are going to formulate supersymmetry using it. To do this we have to consider loop spaces which arise when we express partition functions of supersymmetric theories (Witten indices) using path integrals

$$Z = \text{Str} \exp[-iT\mathcal{H}] = \int_{PBC} \mathcal{D}\Phi \exp(iS[\Phi]) \quad (4)$$

where  $\mathcal{H}$  is the Hamiltonian and  $S[\Phi]$  the corresponding action. All the fields  $\Phi$  are assumed to have periodic boundary conditions in time  $\Phi(x, 0) = \Phi(x, T)$  and to vanish in infinity. The fields are thus elements of the loop space  $L\Phi$ .

The classical model for  $G$ -equivariant cohomology is the Cartan model in which we consider the equivariant exterior derivative for the momentum map  $H = \phi^a H_a$

$$d_H = d - \phi^a \iota_a \quad (5)$$

which squares to  $d_H^2 = -\phi^a \mathcal{L}_a$ . Consequently,  $d_H$  is nilpotent on the  $G$ -invariant subcomplex of differential forms  $\Lambda_G M$  and the equivariant cohomology is

$$H_G^*(M) = H^*(\Lambda_G M) . \quad (6)$$

Previously it has been shown [6] that  $N = 1$  supersymmetric theories can be interpreted in terms of super loop space equivariant cohomology. This formulation uses

the Cartan model. In super loop space the coordinates  $\varphi^k$  can be either Grassmann even or odd and the corresponding 1-forms  $\psi^k$  are thus odd and even, respectively. By introducing auxiliary fields a supersymmetric theory can be written in the form (with implicit space-time integrations)

$$S[\Phi] = S_B + S_F = \int \vartheta_k \dot{\varphi}^k + \psi^k \Omega_{kl} \dot{\psi}^l = (d + \iota_{\dot{\varphi}}) \vartheta = d_{\dot{\varphi}} \vartheta . \quad (7)$$

Here  $\vartheta$  is a pre-symplectic potential in the super loop space and  $\Omega$  the corresponding 2-form. The contraction  $\iota_{\dot{\varphi}}$  is along the vector field that generates the circle action, or the translation in loop parameter  $\tau$ . The supersymmetry means that the action is equivariantly closed

$$d_{\dot{\varphi}}(S_B + S_F) = d_{\dot{\varphi}}^2 \vartheta = \mathcal{L}_{\dot{\varphi}} \vartheta \sim 0 . \quad (8)$$

The Weyl identity

$$d_{\dot{\varphi}}^2 = \mathcal{L}_{\dot{\varphi}} = d/d\tau \quad (9)$$

provides a representation of the supersymmetry algebra. The super loop space structure has been shown to be general by considering the transformation laws of a general  $N = 1$  supermultiplet [6].

To define the Weil and BRST-models for equivariant cohomology we introduce the Weil algebra  $W(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*)$  where  $S(\mathfrak{g}^*)$  is the symmetric algebra on  $\mathfrak{g}^*$  and  $\Lambda(\mathfrak{g}^*)$  the exterior algebra on  $\mathfrak{g}^*$ . The symmetric algebra  $S(\mathfrak{g}^*)$  is generated by commuting basis elements  $\{\phi^a\}$  and  $\Lambda(\mathfrak{g}^*)$  by anticommuting elements  $\{\eta^a\}$ . The Weil differential  $d_W$  acting on  $W(\mathfrak{g})$  is given by

$$\begin{aligned} d_W \eta^a &= \phi^a - \frac{1}{2} f_{bc}^a \eta^b \eta^c , \\ d_W \phi^a &= -f_{bc}^a \eta^b \phi^c . \end{aligned} \quad (10)$$

The Weil differential is nilpotent with a trivial cohomology. The complex of the Weil and BRST-models is the tensor product  $\Lambda M \otimes W(\mathfrak{g})$ .

In the Weil model we introduce the nilpotent differential on  $\Lambda M \otimes W(\mathfrak{g})$

$$d_T = d + d_W . \quad (11)$$

The equivariant cohomology is given by restricting to basic forms which are  $G$ -invariant and horizontal. To do this we introduce an interior product and a Lie derivative on  $W(\mathfrak{g})$  by

$$\begin{aligned} I_a \eta^b &= \delta_a^b , \\ I_a \phi^b &= 0 , \\ L_a &= [d_W, I_a]_+ . \end{aligned} \quad (12)$$

Forms annihilated by  $(\mathcal{L}_a + L_a)$  ( $G$ -invariant) and  $(\iota_a + I_a)$  (horizontal) are elements of the Weil model for  $G$ -equivariant cohomology.

The BRST-model differential is obtained by defining the nilpotent differential on  $W(\mathfrak{g}) \otimes \Lambda M$

$$s = d + d_W + \eta^a \mathcal{L}_a - \phi^a \iota_a . \quad (13)$$

The BRST- and Weil model differentials are related by a conjugation [7]

$$s = \exp(-\eta^a \iota_a) d_T \exp(\eta^b \iota_b) . \quad (14)$$

Since the cohomology of  $d_W$  is trivial the cohomology of  $s$  is the deRham cohomology of  $M$ . The BRST-model for equivariant cohomology is given by the cohomology of  $s$  when it is restricted to basic forms. The reason for calling  $s$  to be a BRST-differential is that eg. in topological Yang-Mills the structure of the BRST operator is exactly the same [4, 5]. The fields  $\eta^a$  are the ghosts for the constraints  $\mathcal{L}_a$  and  $\phi^a$  are the ghosts for ghosts required for first stage reducible theories.

Now we are going to show that a similar interpretation of supersymmetry exists also in the context of the Weil and BRST-models. In fact, the role of the odd coordinates in the super loop space becomes more precise: they are the odd generators  $\eta^a$  of the loop space of the Weil algebra  $LW(\mathfrak{g})$ . Also the auxiliary fields that do not have any dynamics but are introduced to have a balance between the number of even and odd fields are the generators  $\phi^a$  of  $LW(\mathfrak{g})$ . To get dynamics to the fields we add contractions along circle action on  $L\Phi$  and  $LW(\mathfrak{g})$ . The integration over space-time is implicit in the following

$$\begin{aligned} d_T &\rightarrow Q_\tau = d + d_W + \iota_{\dot{\varphi}} + I_{\dot{\eta}} , \\ s &\rightarrow s_\tau = d + d_W - \phi^a \iota_a + \eta^a \mathcal{L}_a + \iota_{\dot{\varphi}} + I_{\dot{\eta}} . \end{aligned} \quad (15)$$

These generators square to

$$Q_\tau^2 = s_\tau^2 = \frac{d}{d\tau} . \quad (16)$$

Here  $\tau$  is the circle parameter which is not necessarily time but can be a light-cone coordinate as well. The relation (16) also provides a representation of the supersymmetry algebra. The operators  $Q_\tau$  and  $s_\tau$  are thus supersymmetry generators. The Weil model generator has the explicit (Lagrangian) realization (now  $\varphi^k$  is even and  $\psi^k \sim d\varphi^k$  odd)

$$Q_\tau = \psi^k \frac{\delta}{\delta \varphi^k} + (\phi^a - \frac{1}{2} f_{bc}^a \eta^b \eta^c) \frac{\delta}{\delta \eta^a} + \dot{\varphi}^a \frac{\delta}{\delta \psi^k} + \dot{\eta}^k \frac{\delta}{\delta \phi^k} . \quad (17)$$

A Hamiltonian realization for the BRST-generator is obtained by introducing the conjugate variables with non-trivial Poisson brackets

$$\begin{aligned} \{p_l, \varphi^k\} &= \{\psi^k, \bar{\psi}_l\} = \delta_l^k , \\ \{\eta^a, \mathcal{P}_b\} &= \{\phi^a, \pi_b\} = \delta_b^a . \end{aligned} \quad (18)$$

The BRST-differential is then

$$s_\tau = \psi^k p_k + (\phi^a - \frac{1}{2} f_{bc}^a \eta^b \eta^c) \mathcal{P}_a - \phi^a \iota_a + \eta^a \mathcal{L}_a + \dot{\varphi}^k \bar{\psi}_k + \dot{\eta}^a \pi_a . \quad (19)$$

Supersymmetric actions can be obtained by acting with  $Q_\tau$  and  $s_\tau$  on a symplectic potential  $\Theta$  on  $LW(\mathfrak{g}) \otimes L\Phi$  since every action of the form  $S = Q_\tau \Theta$  or  $S = s_\tau \Theta$  is automatically supersymmetric in view of (16). However, renormalizability restricts possible symplectic potentials. In the theories that we are going to consider the potential is

$$\Theta = \vartheta_k \psi^k + \frac{1}{2} \phi^a \eta^a - \eta^a W_a \quad (20)$$

where  $W_a$  is the superpotential of the model, as we shall see in specific examples. In the Weil model we obtain the action

$$\begin{aligned} S_W &= Q_\tau \Theta = \int \vartheta_k \dot{\psi}^k + \frac{1}{2} \eta^a \dot{\eta}^a + \frac{1}{2} \Omega_{kl} \psi^k \psi^l \\ &+ \frac{1}{2} \phi^a \phi^a - \frac{1}{2} f_{bc}^a \phi^a \eta^b \eta^c - \phi^a W_a - \eta^a \partial_k W_a \psi^k - \frac{1}{2} f_{bc}^a \eta^b \eta^c W_a . \end{aligned} \quad (21)$$

In the BRST-model a supersymmetric action becomes

$$\begin{aligned} S_B &= s_\tau \Theta = \int \vartheta_k \dot{\psi}^k + \eta^a \dot{\eta}^a - \phi^a H_a + \frac{1}{2} \Omega_{kl} \psi^k \psi^l \\ &+ \frac{1}{2} \phi^a (\phi^a - \frac{3}{2} f_{bc}^a \eta^b \eta^c) - \phi^a W_a - \eta^a \partial_k W_a \psi^k - \eta^a \{H_a, W_b\} \eta^b . \end{aligned} \quad (22)$$

Here we assume that  $\mathcal{L}_a \vartheta = 0$  which allows us to identify  $H_a = \iota_a \vartheta$ . The Weil algebra generator  $\phi^a$  appears purely algebraically in the actions and is therefore an auxiliary field. Also, the interaction that involves the superpotential produces a coupling between  $L\Phi$  and  $LW(\mathfrak{g})$ . In the Weil model we have only trivial circle action in the loop spaces. By choosing suitable Hamiltonians  $H_a$  in the BRST-model we expect to get some non-Abelian generalizations for supersymmetric models with a non-trivial Hamiltonian flow in the loop space. These models might be relevant for non-Abelian localization [10]. Another interpretation [5] for (22) is that it is a constrained system with a constraint algebra (3) which is encompassed by the BRST-operator.

In the supersymmetric models that we consider it is sufficient to use the Abelian Weil model differential with  $f_{bc}^a = 0$ . The Weil model action reduces to

$$\begin{aligned} S_W &= \int \vartheta_k \dot{\psi}^k + \frac{1}{2} \eta^a \dot{\eta}^a + \frac{1}{2} \Omega_{kl} \psi^k \psi^l \\ &+ \frac{1}{2} \phi^a \phi^a - \phi^a W_a - \eta^a \partial_k W_a \psi^k . \end{aligned} \quad (23)$$

An interesting prospect is to consider the BRST-model with an Abelian Weil algebra. In this case the Hamiltonians are in involution and the action

$$\begin{aligned} S_B &= \int \vartheta_k \dot{\psi}^k + \frac{1}{2} \eta^a \dot{\eta}^a - \phi^a H_a + \frac{1}{2} \Omega_{kl} \psi^k \psi^l \\ &+ \frac{1}{2} \phi^a \phi^a - \phi^a W_a - \eta^a \partial_k W_a \psi^k - \eta^a \{H_a, W_b\} \eta^b . \end{aligned} \quad (24)$$

might represent some integrable models.

### 3 The General N=1 Supermultiplet in Four Dimensions

To show that the Weil algebra structure is present in general  $N = 1$  supersymmetric theories in four dimensions we formulate the transformation laws of the general  $N = 1$  supermultiplet [9] using the Weil model supersymmetry generator. The multiplet consists of the following fields: scalar  $M$ , pseudoscalars  $C, N, D$ , vector  $A_\mu$  and Dirac spinors  $\chi, \lambda$ . The transformation properties of the multiplet with a Grassmann spinor parameter  $\zeta$  are

$$\begin{aligned}
\delta C &= \bar{\zeta} \gamma_5 \chi, \\
\delta \chi &= (M + \gamma_5 N) \zeta - i \gamma^\mu (A_\mu + \gamma_5 \partial_\mu C) \zeta, \\
\delta M &= \bar{\zeta} (\lambda - i \not{\partial} \chi), \\
\delta N &= \bar{\zeta} \gamma_5 (\lambda - i \gamma_5 \not{\partial} \chi), \\
\delta A_\mu &= i \bar{\zeta} \gamma_\mu \lambda + \bar{\zeta} \partial_\mu \chi, \\
\delta \lambda &= -i \sigma^{\mu\nu} \zeta \partial_\mu A_\nu - \gamma_5 \zeta D, \\
\delta D &= -i \bar{\zeta} \not{\partial} \gamma_5 \lambda.
\end{aligned} \tag{25}$$

These transformations are generated by the Majorana supercharge

$$Q = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot{\alpha}} \end{pmatrix} \tag{26}$$

which obeys the anticommutator

$$[Q, Q]_+ = 2(\gamma^\mu C) P_\mu. \tag{27}$$

Using the Majorana representation  $\gamma^0 = -\sigma^2 \otimes 1$ ,  $\gamma^1 = -i\sigma^3 \otimes \sigma^1$ ,  $\gamma^2 = i\sigma^2 \otimes 1$ ,  $\gamma^3 = -i\sigma^3 \otimes \sigma^3$  we have the following relevant entries

$$(\gamma^\mu C) P_\mu = \begin{pmatrix} i\partial_+ & * & * & * \\ * & i\partial_+ & * & * \\ * & * & i\partial_- & * \\ * & * & * & i\partial_- \end{pmatrix}. \tag{28}$$

Here  $\partial_\pm$  denote derivatives with respect to light cone coordinates  $x^\pm = x^2 \pm t$ .

For simplicity we choose to consider the supersymmetry transformations generated by the component  $Q_1$ . The following redefinitions of the fields simplify the transformation rules (for details see [6])

$$\begin{aligned}
M' &= M + A_z + \partial_x C, \\
N' &= N + A_x - \partial_z C, \\
\lambda'_1 &= \lambda_1 - \partial_z \chi_1, \\
\lambda'_2 &= \lambda_2 - \partial_x \chi_1, \\
\lambda'_3 &= 2\lambda_3 - \partial_- \chi_1, \\
D' &= D + \partial_x A_z - \partial_z A_x.
\end{aligned} \tag{29}$$

The transformation rules for the primed fields are

$$\begin{aligned}
Q_1 C &= \chi_2 , \\
Q_1(\chi_1, \chi_2, \chi_3, \chi_4) &= (iA_+, i\partial_+ C, iM', iN') , \\
Q_1 M' &= \partial_+ \chi_3 , \\
Q_1 N' &= \partial_+ \chi_4 , \\
Q_1(A_+, A_-, A_x, A_z) &= (\partial_+ \chi_1, -\lambda_3, -\lambda_2, -\lambda_1) , \\
Q_1(\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4) &= (-i\partial_+ A_z, -i\partial_+ A_x, -i\partial_+ A_-, -iD') , \\
Q_1 D' &= -\partial_+ \lambda_4 .
\end{aligned} \tag{30}$$

The exterior derivative and the Weil differential, as well as the contractions can now be read from the transformation laws. The bosonic fields  $\varphi^k \sim (C, A_-, A_x, A_z)$  can be viewed as the coordinates in  $L\Phi$  while the fermionic fields  $\psi^k \sim (\chi_2, -\lambda'_3, -\lambda'_2, -\lambda'_1)$  are their differentials. We thus identify

$$d = \chi_2 \frac{\delta}{\delta C} - \lambda'_3 \frac{\delta}{\delta A_-} - \lambda'_2 \frac{\delta}{\delta A_x} - \lambda'_1 \frac{\delta}{\delta A_z} \tag{31}$$

as the exterior derivative in  $L\Phi$ . The Weil algebra  $LW(\mathbf{g})$  is generated by the fermionic fields  $\eta^a \sim (\chi_1, \chi_3, \chi_4, \lambda'_4)$ . According to the rules they transform to their bosonic superpartners  $\phi^a \sim (iA_+, iM', iN', -iD')$ . This is exactly the transformation generated by the Weil differential  $d_W$  which becomes

$$d_W = iA_+ \frac{\delta}{\delta \chi_1} + iM' \frac{\delta}{\delta \chi_3} + iN' \frac{\delta}{\delta \chi_4} - iD' \frac{\delta}{\delta \lambda_4} . \tag{32}$$

The remaining transformation laws of the multiplet are such that the fields transform to space-time derivatives. These rules are generated by the contractions on  $L\Phi$  and  $LW(\mathbf{g})$

$$\begin{aligned}
\iota_+ &= i\partial_+ C \frac{\delta}{\delta \chi_2} - i\partial_+ A_- \frac{\delta}{\delta \lambda'_3} - i\partial_+ A_x \frac{\delta}{\delta \lambda'_2} - i\partial_+ A_z \frac{\delta}{\delta \lambda'_1} \\
I_+ &= \partial_+ \chi_3 \frac{\delta}{\delta M'} + \partial_+ \chi_4 \frac{\delta}{\delta N'} - \partial_+ \lambda_4 \frac{\delta}{\delta D'} + \partial_+ \chi_1 \frac{\delta}{\delta A_+} .
\end{aligned} \tag{33}$$

The Weil model supersymmetry generator is the sum of these terms

$$Q_+ = d + d_W + \iota_+ + I_+ . \tag{34}$$

It represents the superalgebra since  $Q_+^2 = i\partial_+$ .

## 4 Specific Theories

We shall now show by some examples that  $N = 1$  supersymmetric theories have the Weil algebra structure by explicitly representing the Weil generator and the symplectic



potential. We first concentrate on chiral fields that describe matter fields by considering the supersymmetric quantum mechanics and the four dimensional Wess-Zumino-model. Then we discuss the four dimensional supersymmetric Yang-Mills theory without matter as an example of a model which involves vector superfields.

The supersymmetric quantum mechanics has the action

$$\begin{aligned} S &= \int \frac{1}{2} \dot{q}^2 + \frac{1}{2} (\theta_1 \dot{\theta}_1 + \theta_2 \dot{\theta}_2) - \frac{1}{2} W_q^2 - \theta_2 W_{qq} \theta_1 \\ &= \int \frac{1}{2} \dot{q}^2 + \frac{1}{2} \theta_1 \dot{\theta}_1 + \frac{1}{2} \theta_2 \dot{\theta}_2 + \frac{1}{2} F^2 - \frac{1}{2} F W_q - \theta_2 W_{qq} \theta_1 \end{aligned} \quad (35)$$

where  $W_q = \partial W / \partial q$  is the superpotential and  $F$  an auxiliary field. We notice that the action is of the form (21) with the following identifications. The terms  $\frac{1}{2} \dot{q}^2 \sim \vartheta_k \dot{\varphi}^k$  and  $\frac{1}{2} \theta_1 \dot{\theta}_1 \sim \frac{1}{2} \eta^a \dot{\eta}^a$  are the pre-symplectic potentials on  $L\Phi$  and  $LW(\mathbf{g})$  respectively. The corresponding 2-forms are  $\frac{1}{2} \theta_2 \dot{\theta}_2 \sim \frac{1}{2} \Omega_{kl} \psi^k \psi^l$  and  $\frac{1}{2} F^2$ . The interaction part  $S_{int} = -F W_q - \theta_2 W_{qq} \theta_1 \sim -\phi^a W_a - \eta^a \partial_k W_a \psi^k$  appears as a coupling between the loop spaces  $L\Phi$  and  $LW(\mathbf{g})$ . The exterior derivative on  $L\Phi$  is  $d = \theta_1 \delta / \delta q$  and the Weil differential  $d_W = F \delta / \delta \theta_2$ . The contractions are  $\iota_{\dot{q}} = \dot{q} \delta / \delta \theta_1$  and  $I_{\dot{\theta}_2} = \dot{\theta}_2 \delta / \delta F$ . The Weil model supersymmetry generator is given by

$$Q_t = \theta_1 \frac{\delta}{\delta q} + F \frac{\delta}{\delta \theta_2} + \dot{q} \frac{\delta}{\delta \theta_1} + \dot{\theta}_2 \frac{\delta}{\delta F} \quad (36)$$

and the action is obtained from the symplectic potential

$$\Theta = \frac{1}{2} \dot{\theta}_1 q + \frac{1}{2} F \theta_2 - \theta_2 W_q \quad (37)$$

which coincide with the form in the general discussion.

This generalizes to the four dimensional Wess-Zumino model with the action

$$\begin{aligned} S &= \int \partial_\mu A^* \partial^\mu A - \frac{1}{4} W_A W_{A^*} + \frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi \\ &+ \frac{1}{4} \bar{\psi} (W_{AA} - W_{A^* A^*}) \psi + \frac{1}{4} \bar{\psi} \gamma^5 (W_{AA} + W_{A^* A^*}) \psi \end{aligned} \quad (38)$$

The Majorana representation of the Dirac matrices is  $\gamma^0 = \sigma^1 \otimes 1$ ,  $\gamma^k = -i\sigma^2 \otimes \sigma^k$ ,  $\gamma^5 = \sigma^3 \otimes 1$ . The charge conjugation is  $C = -i\sigma^3 \otimes \sigma^2$  and the Majorana spinor becomes

$$\psi = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \bar{\theta}_2 \\ -\bar{\theta}_1 \end{pmatrix} \quad (39)$$

To write the Wess-Zumino action in the desired form we introduce light-cone coordinates  $x^\pm = x_0 \pm x_3$  and a complex coordinate  $z = x + iy$  and bosonic auxiliary fields  $F, F^*$ . Using these the action can be written as

$$S = \int 2\partial_- A^* \partial_+ A - \frac{1}{2} F F^* - \frac{i}{2} F (W_A + 2\bar{\partial} A) - \frac{i}{2} F^* (W_A + 2\partial A^*)$$

$$\begin{aligned}
& + \quad i\bar{\theta}_1\partial_-\theta_1 + i\bar{\theta}_2\partial_-\theta_2 - i\bar{\theta}_2\partial_+\theta_2 - i\bar{\theta}_2\bar{\partial}\theta_1 - i\bar{\theta}_1\partial\theta_2 \\
& + \quad \frac{1}{2}W_{AA}\theta_1\theta_2 - \frac{1}{2}W_{AA}^*\bar{\theta}_1\bar{\theta}_2
\end{aligned} \tag{40}$$

We choose  $(A, A^*) \sim \varphi^k$  as coordinates and  $(\theta_1, \bar{\theta}_1) \sim \psi^k$  as corresponding 1-forms on  $L\Phi$ . The Weil algebra is generated by coordinates  $\eta^a \sim (\theta_2, \bar{\theta}_2)$  and 1-forms  $\phi^a \sim (F, F^*)$ . We choose the circle action in the light-cone direction  $x^+$ . The Weil model supersymmetry generator is then

$$\begin{aligned}
Q_+ = & \quad i\theta_1\frac{\delta}{\delta A} + i\bar{\theta}_1\frac{\delta}{\delta A^*} + \pi\frac{\delta}{\delta\theta_1} + \pi^*\frac{\delta}{\delta\theta_1} \\
& + \quad \partial_+A\frac{\delta}{\delta\theta_2} + \partial_+A^*\frac{\delta}{\delta\bar{\theta}_2} + i\partial_+\theta_2\frac{\delta}{\delta\pi} + i\partial_+\bar{\theta}_2\frac{\delta}{\delta\pi^*}
\end{aligned} \tag{41}$$

which squares to  $Q_+^2 = i\partial_+$ . The action is obtained from the symplectic potential

$$\Theta = \frac{1}{2}(\partial_-A\bar{\theta}_1 + \partial_-A^*\theta_1) - \frac{1}{2}(\theta_2\pi + \bar{\theta}_2\pi^*) + \frac{i}{2}(\partial A^* + W_A)\theta_2 + \frac{i}{2}(\bar{\partial}A + W_A^*)\bar{\theta}_2. \tag{42}$$

The interpretation of various terms is as in the supersymmetric quantum mechanics.

Finally, the action of the supersymmetric Yang-Mills theory is

$$S = \int -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \frac{i}{2}\bar{\psi}\gamma^\mu D_\mu\psi \tag{43}$$

where  $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc}A_\mu^b A_\nu^c$  is the field strength of the gauge field  $A_\mu^a$ ,  $\psi^a$  is a Majorana spinor in the adjoint representation of the gauge group and  $D_\mu^{ab} = \delta^{ab}\partial_\mu + f^{abc}A_\mu^c$  is the covariant derivative. To write this action in the required form we first consider the Hamiltonian form of the action

$$S = \int E_i^a \dot{A}_i^a - \frac{1}{2}E_i^a E_i^a - \frac{1}{2}B_i^a B_i^a + \frac{i}{2}\bar{\rho}^a \dot{\rho}^a + \frac{i}{2}\bar{\eta}^a \dot{\eta}^a + \rho^a \sigma^i D_i^{ab} \eta^b \tag{44}$$

which is subject to the Gauss law  $\mathcal{G}^a = D_i^{ab}E_i^b + \frac{i}{2}f^{abc}(\rho^\dagger{}^b \rho^c + \eta^\dagger{}^b \eta^c) \sim 0$ . Here we have defined the spinors

$$\rho^a = \begin{pmatrix} \theta_1^a \\ \bar{\theta}_1^a \end{pmatrix}, \quad \eta^a = \begin{pmatrix} \theta_2^a \\ \bar{\theta}_2^a \end{pmatrix}. \tag{45}$$

The change the variables  $E_i^a \rightarrow E_i^a + \dot{A}_i^a$  transforms the action to a form which admits an interpretation in terms of the Weil model:

$$S = \int \frac{1}{2}(\dot{A}_i^a)^2 - \frac{1}{2}(E_i^a)^2 + iE_i^a B_i^a + \frac{i}{2}\rho^\dagger{}^a \dot{\rho}^a + \frac{i}{2}\eta^\dagger{}^a \dot{\eta}^a + \rho^a \sigma^i D_i^{ab} \eta^b. \tag{46}$$

The term  $\frac{1}{2}(\dot{A}_i^a)^2 \sim \vartheta_k \dot{\varphi}^k$  is the pre-symplectic potential. The electric field  $E_i^a \sim \phi^a$  is an auxiliary field. The second term is therefore the symplectic 2-form on  $LW(\mathfrak{g})$ .

The symplectic 1-forms on  $L\Phi$  and  $LW(\mathfrak{g})$  are thus  $\rho^a \dot{\rho}^a$  and  $\eta^a \dot{\eta}^a$ , respectively. The magnetic field  $B_i^a = \frac{1}{2} \epsilon_{ijk} F^{ajk}$  appears as the superpotential in this approach.

According to our standard recipe we can write the Weil model supersymmetry generator, which in this case is spinorial. The circle action is parameterized by the time:

$$Q_t^\beta = \rho_\alpha^a \sigma_{\alpha\beta}^i \frac{\delta}{\delta A_i^a} + E_i^a \sigma_{\alpha\beta}^i \frac{\delta}{\delta \eta_\alpha^a} + \frac{3i}{2} \dot{A}_i^a \sigma_{\alpha\beta}^i \frac{\delta}{\delta \bar{\rho}_\alpha^a} + \frac{2i}{3} \dot{\eta}_\alpha^a \sigma_{\alpha\beta}^i \frac{\delta}{\delta E_i^a} . \quad (47)$$

Here  $\sigma^i$  are the Pauli matrices. The action is then obtained from the symplectic potential of the form (20)

$$\Psi_\delta = \frac{i}{6} \bar{\rho}_\gamma^c \sigma_{\gamma\delta}^k \dot{A}_k^c - \frac{1}{4} \eta_\gamma^c \sigma_{\gamma\delta}^k E_k^c + \frac{i}{2} \eta_\gamma^c \sigma_{\gamma\delta}^k B_k^c . \quad (48)$$

## 5 Conclusions

In conclusion, we have applied Weil model for equivariant cohomology to give a geometrical interpretation for  $N = 1$  supersymmetry. In the Lagrangian formulation we have shown that supersymmetry can be interpreted in terms of bosonic loop space equivariant cohomology and Abelian Weil algebras that are introduced in the Weil model. In this formulation half of the fermionic fields were interpreted as differentials  $\psi^k$  for half of the bosonic fields  $\varphi^k$ . The other half of the fermions were interpreted as coordinates  $\eta^a$  on the Weil algebra  $LW(\mathfrak{g})$ . Their superpartners, the auxiliary fields, were the one forms  $\phi^a$  on  $LW(\mathfrak{g})$ . This identification made the role of anticommuting coordinates and commuting 1-forms in the super loop space quite clear: they are the odd and even generators of the Weil algebra, respectively. The dynamics of the fields was given by contractions along circle actions in the loop spaces.

We also discussed some aspects of the BRST-model for equivariant cohomology. The BRST-differential has the structure of a BRST-operator in constrained systems. We thus propose that many supersymmetric constrained systems can be realized by our formalism. Non-Abelian Weil algebras with Hamiltonians  $H_a$  could produce non-trivial Hamiltonian flow in the loop space. This might give some interesting generalizations for supersymmetric theories. With Hamiltonians  $H_a$  in involution the systems might be integrable.

The Weil algebra structure was shown to be general in supersymmetry by considering the transformation rules of the general  $N = 1$  supermultiplet. As examples we analyzed supersymmetric quantum mechanics, and four dimensional Wess-Zumino and super Yang-Mills theories with the Weil model.

Some prospects are in developing our construction to extended supersymmetric theories and integrable models. It would also be interesting to consider theories in which we have non-Abelian Weil algebra structures instead of Abelian which we have discussed. This might be relevant for non-Abelian localization of path integrals.

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